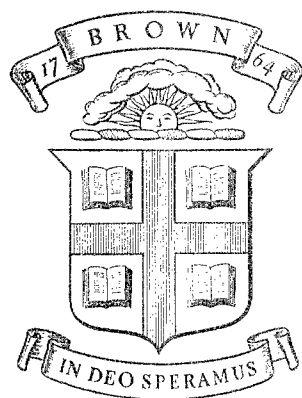


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ELASTIC-PLASTIC ANALYSIS OF  
PRESSURE VESSEL COMPONENTS

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# Elastic-Plastic Analysis of Pressure Vessel Components

by

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## ABSTRACT

Survey on the use of digital computers for elastic-plastic analysis of pressure vessel components. Review of linear incremental stress strain relations for a strain hardening Prandtl-Reuss material with a von Mises yield criterion, formation of generalized stress strain relations. Case studies of axisymmetric elastic-plastic analysis of a torispherical pressure vessel, a flush cylindrical nozzle in a sphere and a thick-walled cylinder under internal pressure.

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## Introduction

Elastic-plastic problems have traditionally been one of the more difficult problems to solve. This was due to the analytical difficulties caused by a moving elastic-plastic boundary and a nonlinear stress strain relation. The earlier solutions were mostly confined to elastic-perfectly plastic analysis of problems with simple geometry and axial symmetry such as thick-cylinders [1-3] under internal pressure. Allen and Southwell [4], by using a stress function for the elastic and elastic-plastic region were able to find relaxation solutions to two-dimensional elastic-plastic problems.

The advent and widespread use of the digital computer has recently led to the solution of many important elastic-plastic problems. Many of these problems have a bearing on and were originally formulated in connection with pressure vessels and other piping components.

Over the past decade two distinct methods have been evolved to solve the elastic-plastic problem. Mendelson and Manson [5] introduced the method of thermal or initial strains using the deformation theory of plasticity. In this approach, the equilibrium equations and the plastic strains are solved for in an iterative manner. The effect of the plastic straining is brought in as a pseudo-load on the right-hand side of the equilibrium equations. This method was first applied to shells of revolution by Stern [6]. Spera [7] extended the analysis to account for discontinuous shells by using a more general finite difference method. The initial strains method was subsequently modified by Stern [8] and Mendelson [9] to use the incremental theory of plasticity. The method has been widely used in conjunction with finite element analysis [10-13]. Convergence problems have been encountered with this method in its differential equation formulation. Mendelson [9] in an alternative formulation in terms of strains, has found improved convergence.

The other approach to the elastic-plastic problem will be referred to here as the tangent modulus method. The writer and his colleagues [14-18] have also referred to it as the "stiffness method" but in view of its use with the direct stiffness method in finite element analysis, the continuation of the latter terminology could cause confusion.

The essence of the tangent modulus method is to obtain a linear stress strain relation for an increment of load. The full analysis is carried out in increments and the elastic-plastic body behaves as though it were made of a piecewise linear anisotropic elastic material. The tangent modulus method has recently been combined with the finite element method [19-22]. With reference to pressure vessel and piping components. Marcal and King [21] have applied the tangent modulus method to study axi-symmetric solid bodies of revolution and Khojasteh-Bakht [22] has used the method in an analysis of the axi-symmetric shell of revolution.

Marcal [23] has noted the close relationship between the two methods of elastic-plastic analysis and has shown that both methods can be based on the linear incremental stress-strain relations for an elastic-plastic material.

Because of its importance, the formulation of a linear incremental stress-strain relation will be reviewed in this survey. This relation is then used to form piecewise linear generalized stress strain relations for plates and shells. The theoretical discussion is then followed by case studies of three examples relevant to pressure vessel design.

### Theoretical Considerations

In this section we shall establish the linear incremental stress strain relation for an elastic-plastic material. The material behavior is governed by the incremental theory of plasticity and the von Mises yield criterion. It is assumed to work-harden according to an isotropic strain hardening criterion.

Earlier work resorted to the inversion of a matrix because of the need to avoid division by zero in the case of an elastic-perfectly plastic material [15]. However, recent formulations [24,25] have circumvented this and allow the linear incremental relations to be obtained in closed form for all cases of elastic-plastic behavior.

We shall adopt matrix notation for the following formulation of the stress strain relations. The plastic increment of strain  $\{de_p\}$  is given by the normal flow rule of plasticity,

$$\{de_p\} = d\bar{e}_p \left\{ \frac{\partial \bar{\sigma}}{\partial \sigma} \right\} \quad (1)$$

where  $d\bar{e}_p$  is the equivalent plastic strain

$\bar{\sigma}$  is the equivalent yield stress

$\{\sigma\}$  is the stress vector

and the prefix  $d$  is used to denote an increment.

The von Mises yield criterion is now written in incremental form

$$\left[ \frac{\partial \bar{\sigma}}{\partial \sigma} \right] \{d\sigma\} = d\bar{\sigma} = H' d\bar{e}_p \quad (2)$$

where  $H'$  is the slope of the equivalent stress equivalent strain curve and  $[ ]$  is used to denote a row vector. Because the elastic components of strains are the only strains that can be associated with changes in stresses, the increment of stress is related to the increment of strain by

$$\{d\sigma\} = [S] \{de_e\} = [S](\{de\} - \{de_p\}) \quad (3)$$

where  $[S]$  is the elastic strain to stress transformation matrix

$\{de\}$  is the total increment of strain

and the subscript  $e$  is used to denote the elastic component of strain.

By multiplying (3) by  $\left[\frac{\partial \bar{\sigma}}{\partial \sigma}\right]$  and using (1) and (2), we obtain an expression for the equivalent plastic strain increment  $d\bar{e}_p$

$$d\bar{e}_p = \frac{\left[\frac{\partial \bar{\sigma}}{\partial \sigma}\right] [S] \{de\}}{H' + \left[\frac{\partial \bar{\sigma}}{\partial \sigma}\right] [S] \left\{\frac{\partial \bar{\sigma}}{\partial \sigma}\right\}} \quad (4)$$

Substituting for the equivalent plastic strain in (3) and rearranging, we obtain the required linear incremental stress strain relation

$$\{d\sigma\} = \left([S] - \frac{[S] \left\{\frac{\partial \bar{\sigma}}{\partial \sigma}\right\} \left[\frac{\partial \bar{\sigma}}{\partial \sigma}\right] [S]}{H' + \left[\frac{\partial \bar{\sigma}}{\partial \sigma}\right] [S] \left\{\frac{\partial \bar{\sigma}}{\partial \sigma}\right\}}\right) \{de\} \quad (5)$$

The term<sup>\*</sup> in the bracket of (5) may be interpreted as the required correction to the elastic stress strain relation which keeps the stress increment on the expanding yield surface (or tangential to the yield surface in the case of an elastic-perfectly plastic material). An examination of the numerator of the second term in the bracket shows that the term is symmetric. This linear relation is, of course, the same relation as that found previously in the earlier works [15,19,20].

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<sup>\*</sup>The matrix  $[p^-]$  first introduced in [23] will be used to refer to this term in subsequent discussion.

### Note on the Transition Region

Because the solution takes place in increments a small region adjacent to the elastic-plastic interface starts out as elastic and ends up by being elastic-plastic. This has been called the transition region. By suitably weighting the elastic and the elastic-plastic stress strain relations, it has been found possible to reduce the number of increments required for a full elastic-plastic solution [21]. If  $m$  is the proportion of the strain increment required to cause yield during that increment, the weighted stress strain relation for the transition region becomes

$$\{d\sigma\} = \left( [S] - \frac{(1-m)[S] \left\{ \frac{\partial \bar{\sigma}}{\partial \sigma} \right\} \left| \frac{\partial \bar{\sigma}}{\partial \sigma} \right| [S]}{H' + \left| \frac{\partial \bar{\sigma}}{\partial \sigma} \right| [S] \left\{ \frac{\partial \bar{\sigma}}{\partial \sigma} \right\}} \right) \{de\} \quad (6)$$

### Piecewise Linear Generalized Stress Strain Relations

The linear relation (5) can now be used to form linear generalized stress strain relations for a plate or a shell by integrating through the thickness.

The strain increment at a point in a shell is given by the sum of the mid-wall and bending components.

$$\{de\} = \{d\bar{e}\} + z\{dk\} \quad (7)$$

where  $\{d\bar{e}\}$  is the mid-wall component of the strain increment

$\{dk\}$  is the bending component of the strain increment, and

$z$  is the distance from the center of the shell wall

The increment of the direct  $\{dN\}$  and bending  $\{dM\}$  stress resultants are given by integrating through the thickness

$$\left\{ -\frac{dN}{dM} \right\} = \int_{-H}^H \left\{ \frac{d\sigma}{z d\sigma} \right\} dz \quad (8)$$

where  $H$  is the half-wall thickness. Substituting (5) and (7) in (8), we obtain

$$\left\{ -\frac{dN}{dM} \right\} = \int_{-H}^H \begin{bmatrix} [p^-] & [p^-]z \\ [p^-]z & [p^-]z^2 \end{bmatrix} dz \left\{ -\frac{d\bar{e}}{d\bar{k}} \right\} \quad (9)$$

The shell wall is divided into a number of stations through the thickness (say 11). The stress history is kept for each of these stations. The matrix  $[p^-]$  is evaluated at the start of each increment for each station and the matrix of equation (9) is formed by numerical integration through the thickness. The actual manner of using equation (9) in an elastic-plastic shell solution depends on the type of shell theory used, i.e., to say whether it results in a differential equation or a matrix equation as in the finite element method. In the case of a differential equation formulation, terms exist which require that equation (9) be further differentiated by some independent variable of length (see, for instance, [15]). In either case the substitution is straightforward and it is not proposed to enter into details here. The interested reader is referred to the references already cited.

### Case Studies

In this section we illustrate the kinds of analysis that can be performed by existing elastic-plastic computer programs. The examples are taken from work with which the writer has been associated because of his greater familiarity with the material. Other results can, of course, be found in the references already given.



### 1. Torispherical Pressure Vessel

The mild-steel torispherical vessel shown in Fig. 1 was tested by Stoddart and Owen [26] and analyzed in the elastic perfectly plastic region by Marcal and Pilgrim [16]. Figure 1 also shows the elastic strain distribution. Good agreement was found between theory and experiment. Figure 2 gives a comparison of the elastic-plastic strains at the position of maximum meridional strain (45° station). Good agreement was again found between theory and experiment. Figure 2 also shows an interesting feature of most elastic-plastic shells in bending and that is the small value of strain reached at the maximum pressure. This point should be borne in mind when materials of construction are being chosen. Figure 3 shows the progress of the yielding throughout the shell. Loads are marked next to the corresponding elastic-plastic interface. The loads are given in dimensionless form where 1 unit is equal to the load at first yield.

### 2. Flush Nozzle with Fillet Weld

The second example considers the behavior at the junction of a flush cylindrical nozzle with a spherical shell. It shows that additional considerations must be introduced in studying shell intersection problems. The theoretical work of Marcal and Turner [17] followed the thesis of O'Connell and Chubb [27] that the line loads caused by an adjoining shell should be spread over a band. The basic difference between the concept of a simple shell theory and the equivalent band replacement theory can best be understood by referring to Fig. 4. Figure 4a shows a flush nozzle with a fillet weld. The dimensions of the lengths of the zones,  $a$  and  $b$  are marked on the cylinder (shell A) and the sphere (shell B);  $a$  and  $b$  are interpreted as the lengths along the shell which are attached to the thickness of the adjacent shell. Figure 4b and 4c shows the shell mid-radii where the shell section appears as a line. In Fig. 4b the concentrated line loads are shown acting at

the junction. The bands of equivalent pressures are shown in Fig. 4c. The radial force  $F$  has been replaced by a radial pressure  $p_F = F/a$  on shell A, but remains as a line load  $F$  acting along the center-line of shell B. Similarly, the axial force  $W$  has been replaced by an axial pressure  $p_W = W/b$  on shell B whilst remaining as a line load  $W$  acting along the center-line of shell A.

Flush nozzles with fillet welds were tested by Dinno and Gill [28]. The dimensions for two of these nozzles are shown in Table 1 while the results are shown in Figs. 5 and 6. Figure 5 gives the pressure vertical displacement curve at the junction. The results from the equivalent band replacement theory are shown as full lines while the results of the simple shell theory are in dotted lines. Experimental limit pressures obtained by Dinno and Gill are also shown. Figure 6 shows the pressure maximum meridional strain curves. Finally, Table 2 summarizes the experimental results obtained for the two vessels. Agreement between the band theory and experiment is good and the necessity of using an equivalent band theory instead of the simple shell theory is clearly established. Similar results with a less striking difference between the two shell theories were observed for the flush nozzles without fillet welds tested by Cloud [29].

### 3. Thick Cylinder with Internal Pressure

The final example is that of a thick cylinder under internal pressure. The assumption of a plane strain end condition was made. The thick pressure vessel was analyzed by the finite element method using the tangent modulus method [21]. The finite element analysis was made by subdividing the vessel into rings with triangular cross sections as suggested by Clough and Rashid [30]. This idealization is shown in Fig. 7. Figure 8 gives a comparison between the results for the finite element method and a numerical procedure developed by

Hodge and White [3]. The cylinder studied had a 2 to 1 diameter ratio and an elastic-perfectly plastic material was assumed.

#### Conclusions and Recommendations for Future Work

Linear incremental stress strain relations exist for a Prandtl-Reuss material with a von Mises yield criterion. The linear incremental relations can be used to form piecewise linear generalized stress strain resultants. This reduces the difficulty of performing an elastic-plastic analysis to a similar level as that of performing a series of anisotropic elastic analyses. The next theoretical advance would appear to be the inclusion of the linear stress strain relations in a large displacement shell analysis.

A few computer programs already exist for the elastic-plastic analysis of axi-symmetric pressure vessels and its components. Many others are under active development. These programs are being developed at such a rate and in so general a form that there is a risk that our understanding of elastic-plastic pressure vessel behavior may be outstripped by our analytical ability. In particular, much work remains to be done in understanding the behavior of end-closures of thick pressure vessels. Here the use of the finite element analysis can contribute towards the study of threaded end plugs as well as torispherical end-closures.

Finally, it is relevant to point out that the translation of the recent theoretical and experimental advances into meaningful design codes has yet to be pursued.

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Table 1. Test vessel data from reference [28]  
all dimensions in inches

| Nozzle                         | $D_1$  | $D_5$  |
|--------------------------------|--------|--------|
| Branch radius, $r$             | 2.875  | 2.875  |
| Branch thickness, $t$          | 0.125  | 0.125  |
| Sphere radius, $R$             | 8.843  | 8.812  |
| Sphere thickness, $T$          | 0.312  | 0.250  |
| $t/T \div r/R$                 | 0.4    | 0.5    |
| $r/T$                          | 23.0   | 23.0   |
| $R/T$                          | 28.34  | 35.25  |
| Thickness + fillet $a$         | 0.812  | 0.425  |
| Thickness + fillet $b$         | 0.710  | 0.545  |
| Yield stress, $\text{lb/in}^2$ |        |        |
| Branch                         | 44.000 | 44.300 |
| Sphere                         | 39.200 | 38.800 |

Table 2. Limit of proportionality and limit load pressures All pressures in lb/in<sup>2</sup>.

| Nozzles                                | D <sub>1</sub> | D <sub>5</sub> |
|--|----------------|----------------|
| Limit of proportionality:              |                |                |
| From experimental deflections          | 800            | 800            |
| Present theory                         | 740            | 630            |
| Simple shell theory                    | 400            | 340            |
| Estimated limit load:                  |                |                |
| Experimental                           | 1360           | 1180           |
| Present theory at 1.5 per cent strain) | 1320           | 1070           |
| Present theory at instability          | 1350           | 1135           |
| Limit analysis $7p/8$                  | 1110           | 855            |
| Limit analysis $p$                     | 1270           | 980            |
| Simple shell theory                    | 508            | 505            |

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FIGURE CAPTIONS FOR PAPER BY MARCAL

- Fig. 1 Stress distribution for torispherical vessel, internal pressure 100 lb/in<sup>2</sup>.
- Fig. 2 45° Station surface strains.
- Fig. 3 Progressive yielding.
- Fig. 4 Shell junction representations.
- Fig. 5 Pressure displacement curves for flush nozzle.
- Fig. 6 Pressure maximum meridional strain curves for flush nozzle.
- Fig. 7 Rotated triangular element.
- Fig. 8 Pressure surface strain curves for thick cylinder.

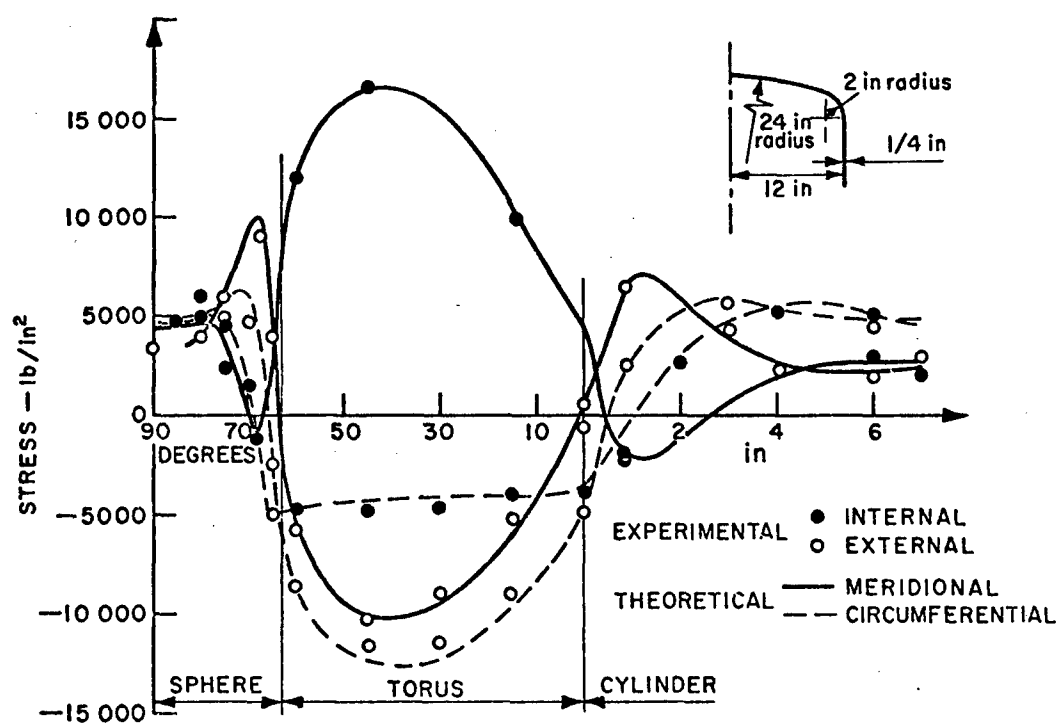


FIGURE 1

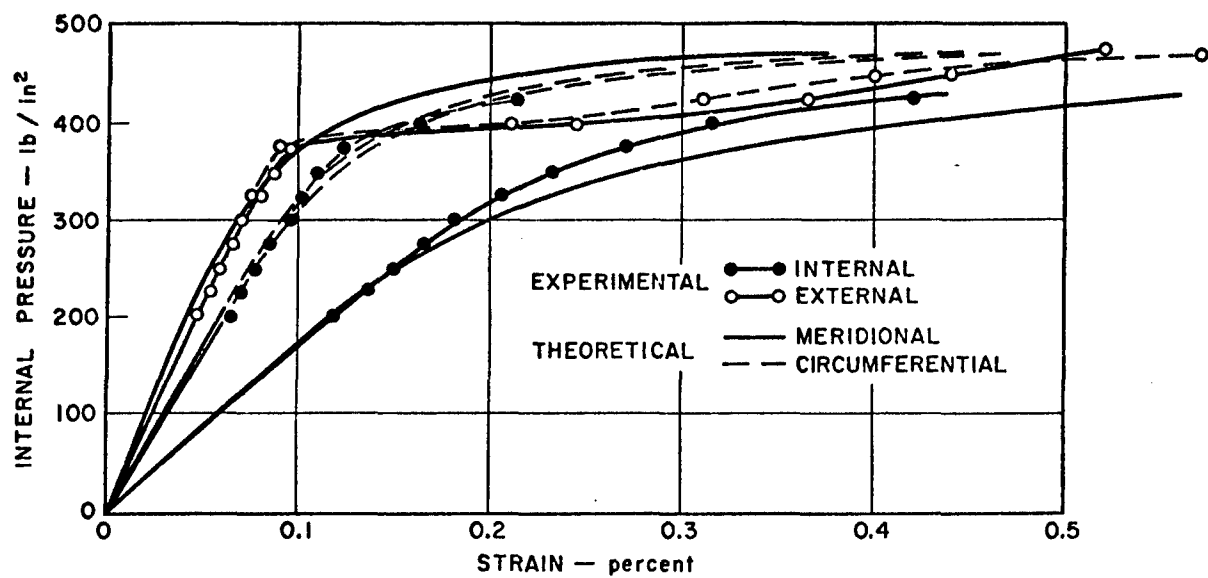


FIGURE 2

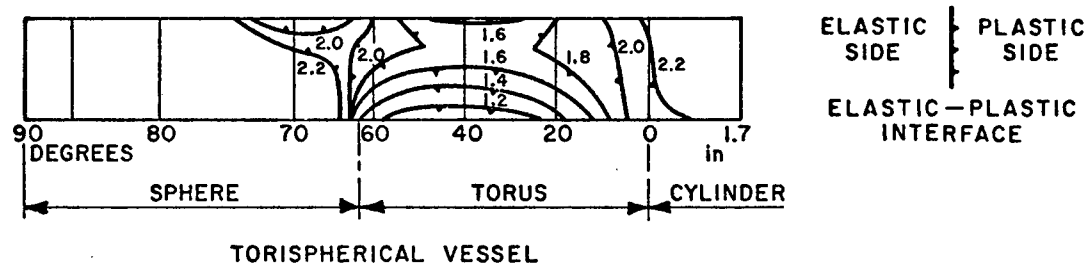


FIGURE 3

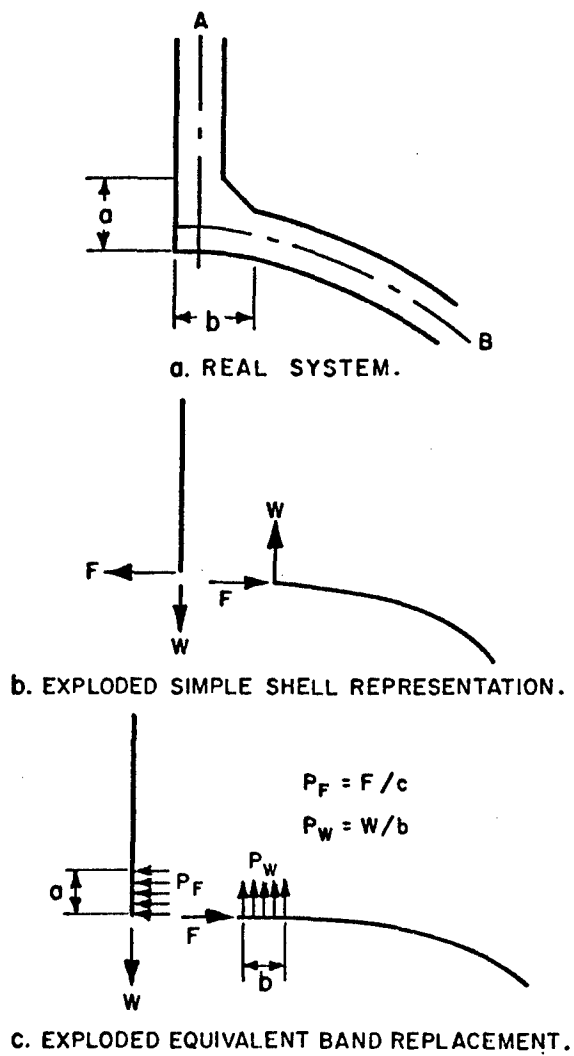


FIGURE 4

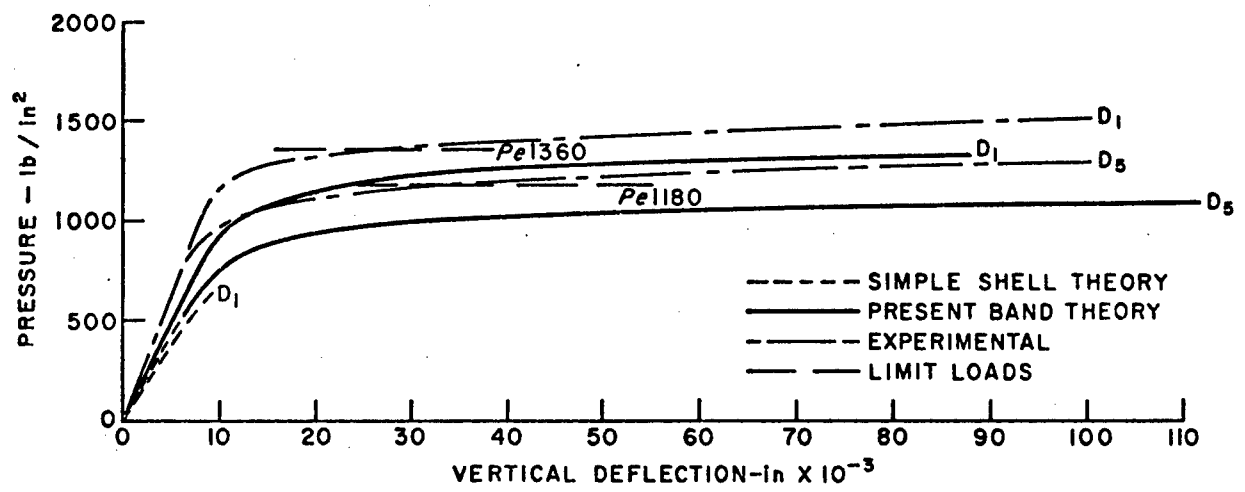


FIGURE 5

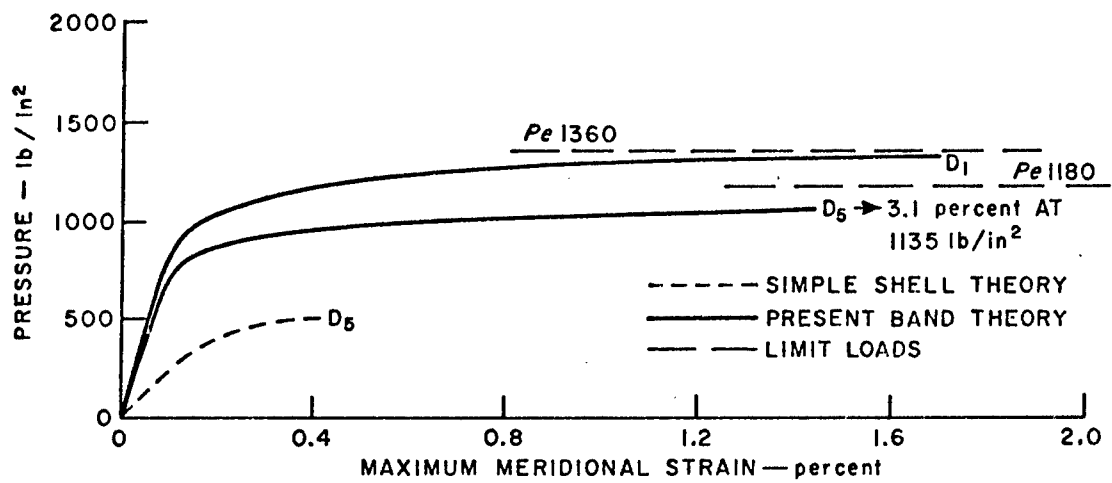


FIGURE 6

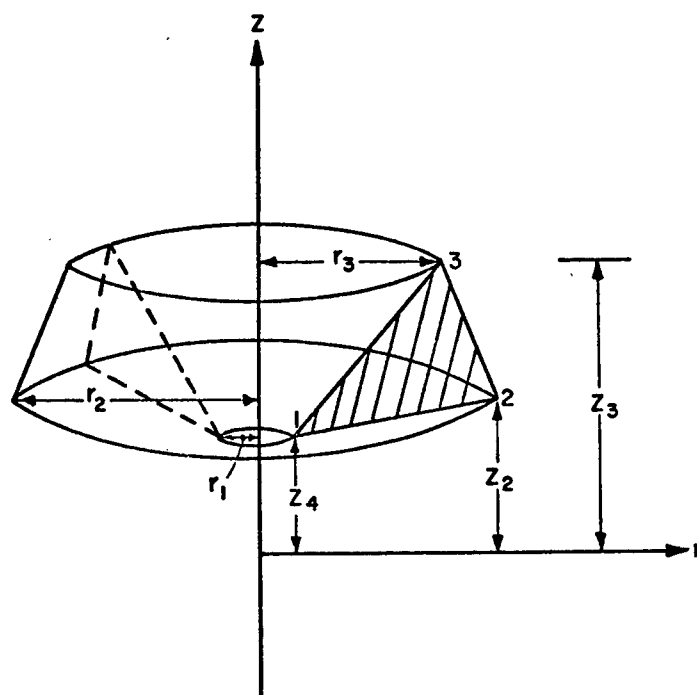


FIGURE 7



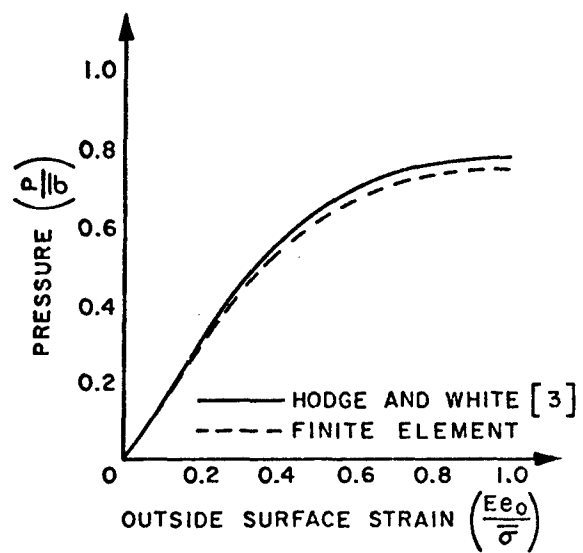


FIGURE 8